Measure-preserving PL dynamical systems in $\mathbb{R}^3$ with bounded trajectories

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Abstract

We will demonstrate here the construction of a piecewise-linear, measure-preserving, non-singular dynamical system on $\mathbb{R}^3$, with each trajectory contained in a bounded set. This is achieved by creating a nested sequence of *PL* approximations of solid tori, whose union is $\mathbb{R}^3$. We then build a 1-foliation of each of the nested tori, which is modified using the slanted suspension construction of [GK],[GKK], to become a measured foliation of $\mathbb{R}^3$, with each leaf contained in a bounded set. Finally, the leaves in this measured foliation are used to create a *PL*, measure-preserving, non-singular, dynamical system, with each orbit contained in a bounded set.

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- All functions are assumed to be *PL* homeomorphisms, meaning that they are linear on each simplex in our triangulation.

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- A *continuous dynamical system* is a triple $(\mathbb{R}, \Omega, \pi)$, with Ω a topological space and $\pi : \mathbb{R} \times \Omega \to \Omega$, such that
  - π is continuous
  - For any $x \in \Omega$, and any $t_1, t_2 \in \mathbb{R}$, we have $\pi(0, x) = x$ and $\pi(t_1, \pi(t_2, x)) = \pi(t_1 + t_2, x)$.

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- A point $x \in \Omega$ is a *fixed point* if $x = tx$ for all $t \in \mathbb{R}$.
- $(\mathbb{R}, \Omega, \pi)$ is a *non-singular* dynamical system if it contains no fixed points.

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- If μ is a measure on Ω, such that for any measurable set $A \subset \Omega$ and any $t \in \mathbb{R}$, $\mu(A) = \mu(\pi^{-1}(t, A))$, then $(\mathbb{R}, \Omega, \pi, \mu)$ is a *measure-preserving* dynamical system [Ro].

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- Since our manifolds are all *PL*, we assume our measures are *simplicial*, that is, the measure is equivalent to Lebesgue measure on each simplex (after embedding into $\mathbb{R}^n$ under a chart map.)

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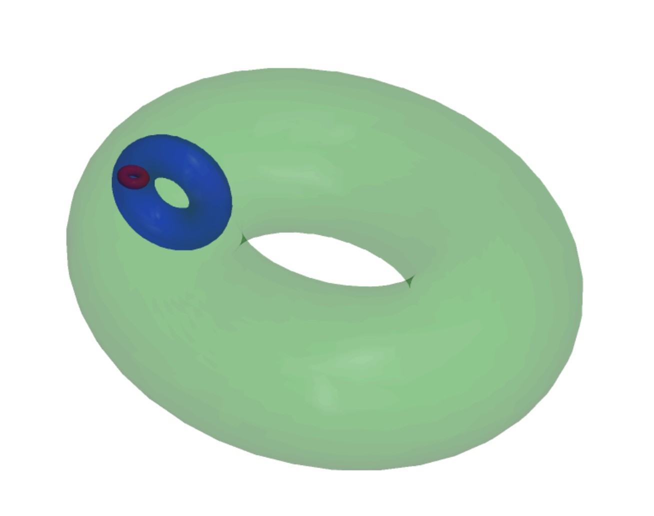

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- Let T_n^{PL} be a *PL* torus, in the *xy*-plane if n is even, and the *yz*-plane if n is odd, centered at $(0,(4)12^n,0)$, with minor radius $m_n = \frac{5}{2}M_{n-1}$ and major radius $M_n = 12M_{n-1}$. Again, rotate by $\pi/16$ after each construction.

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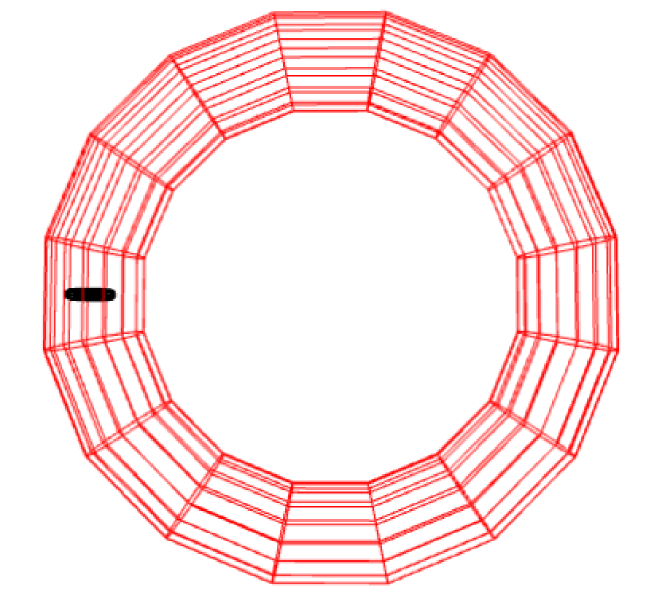

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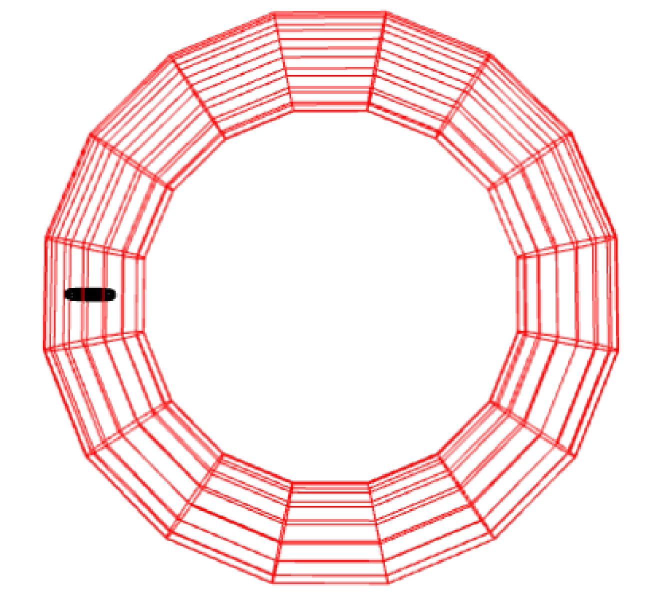

1-foliations

Let M be an n-manifold. Fix some indexing set A. Let $\mathcal{F} = \{L_\alpha : \alpha \in A\}$ be a collection of arcwise connected subsets of M. $\mathcal{F}$ is a *1-dimensional foliation of M* if

(i) $L_\alpha \cap L_\beta = \emptyset$ for $\alpha \neq \beta$

(ii) $\bigcup_{\alpha \in A} L_\alpha = M$.

(iii) Given any point $p \in M$, there exists a chart of $(U_\lambda, \varphi_\lambda)$ about p, such that for L_α with $L_\alpha \cap U_\lambda \neq \emptyset$, each path component of $\varphi(L_\alpha \cap U_\lambda)$ is of the form

$$\{x_1 \in \varphi_\lambda(U_\lambda) : x_2 = c_1, x_3 = c_2, \ldots, x_n = c_{n-1}\}$$

where each c_i is a constant determined by L_α.

Each L_α is a *leaf* of the foliation $\mathcal{F}$. We can view the embeddings as splitting $\mathbb{R}^n$ into two pieces, $\mathbb{R}$ and $\mathbb{R}^{n-1}$. On $\mathbb{R}$, the coordinates of the embedding vary with L_α, but on $\mathbb{R}^{n-1}$, the coordinates are fixed.

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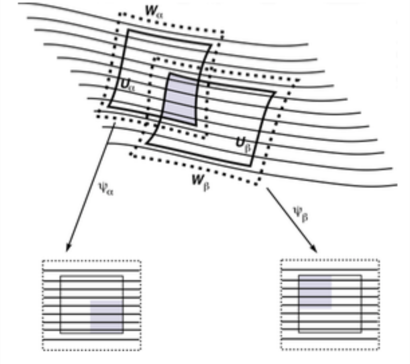

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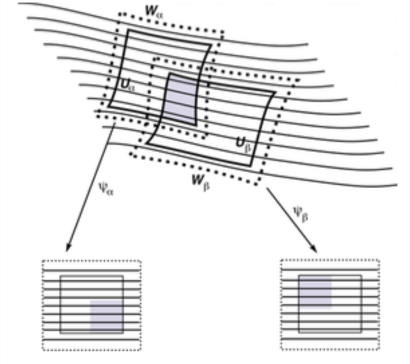
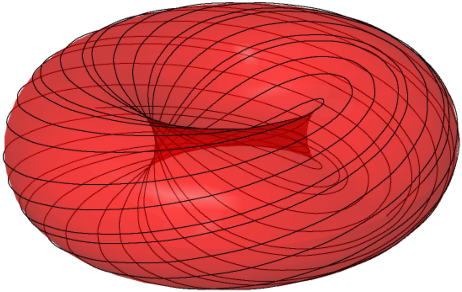


[H, LN] We say the foliation is *oriented* if the embedding preserves the usual orientation of $\mathbb{R}$ [T].

Foliated torus



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Begin with some manifold M and a 1-foliation $\mathcal{F}$.

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- T is *small* if it can be surrounded by a single flow box.
- A *transverse measure* μ on $\mathcal{F}$ is a function which assigns each small transversal a finite non-negative number[RS].

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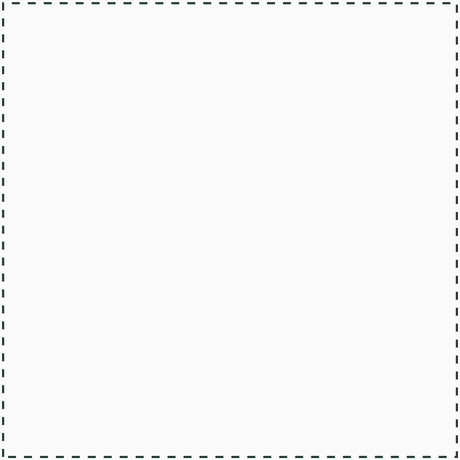
- Given two transversals, α, β, a measure μ is *invariant* if, when α and β are isotopic, with isotopy parallel to the foliation, then $\mu(\alpha) = \mu(\beta)$.
- Then the pair $(\mathcal{F}, \mu)$ is a *measured foliation*. [FLP]

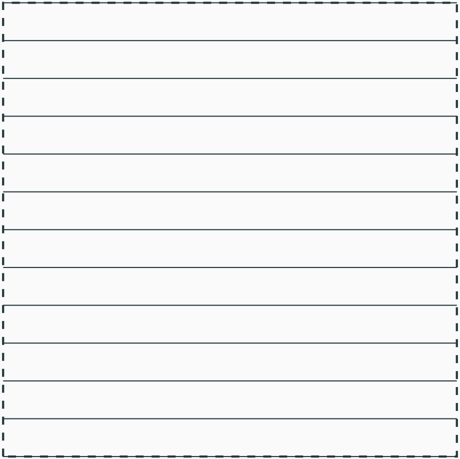
Example

Start with a flow box



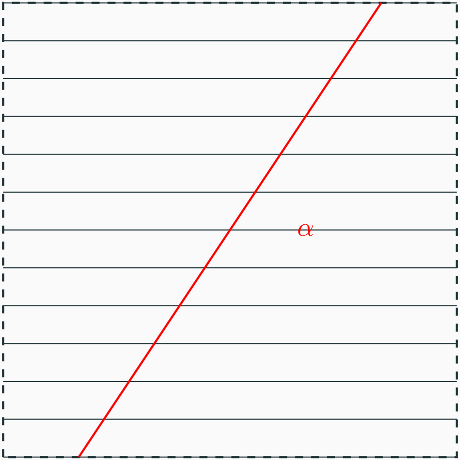
Example

Only one coordinate changes on each leaf.



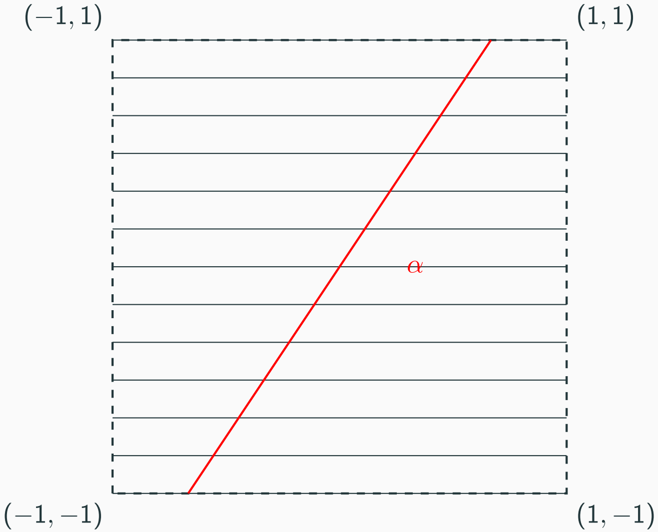
Example

Add in a transversal, which we call α.

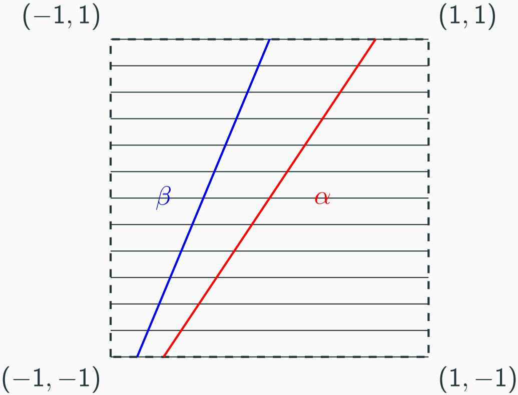


Example

Add coordinates, and define $\mu(\alpha) = 2$



Isotopies of transversals



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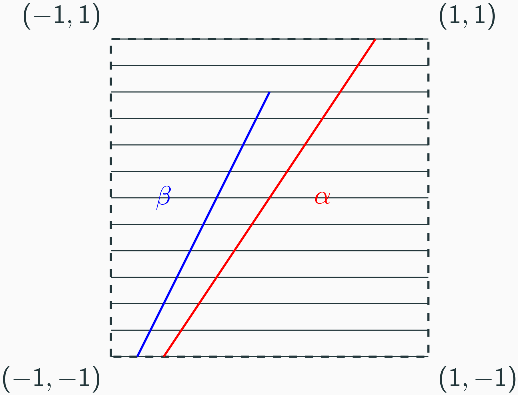
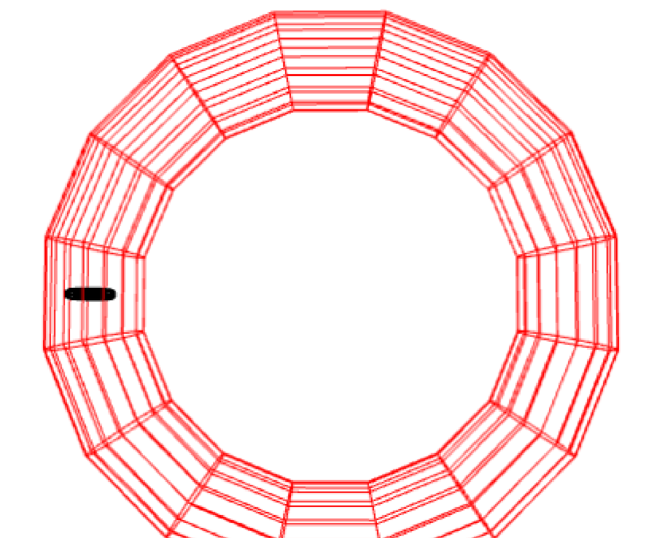


From foliations to dynamics

We can describe how to foliate our nested tori, but we need to be sure we can end up with a dynamical system to satisfy our theorem.



Hyperboloid Example

Let M be the hyperboloid of one sheet in $\mathbb{R}^3$, given by the parametric equations

$$x = \sqrt{u^2 + 1}\cos(v)$$
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$$z = u$$

for $u \in [-1, 1]$ and $v \in [0, 2\pi]$.

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Assume M is foliated by circular leaves, lying parallel to the xy-plane

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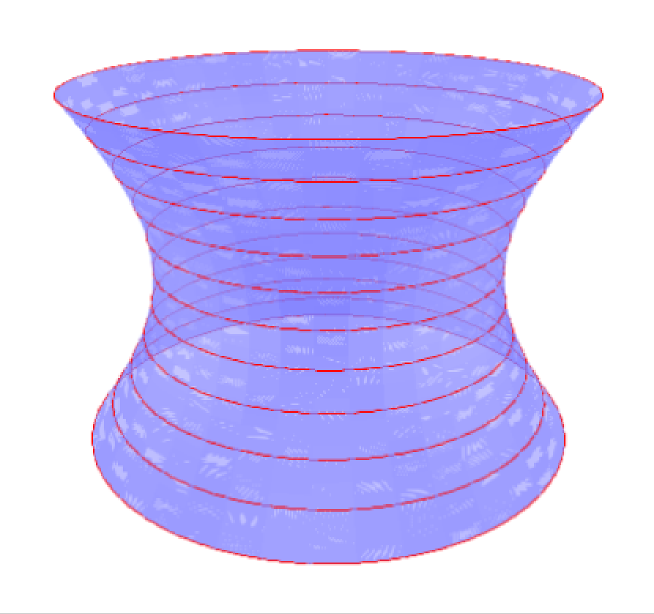

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- Since our leaves are circular and parallel to the xy-plane in $\mathbb{R}^3$, when the chart map is applied, the change in the y coordinate on each small transversal that makes up α will be kept constant during an isotopy parallel to the leaves that moves α to β. Therefore, $\mu(\alpha) = \mu(\beta)$.

Isotopic transversals

To give a specific example, let α be the transversal connecting the points $(\sqrt{2}, 0, 1)$ and $(\sqrt{2}, 0, -1)$, and β connect the points $(0, \sqrt{2}, 1)$ and $(0, \sqrt{2}, -1)$.

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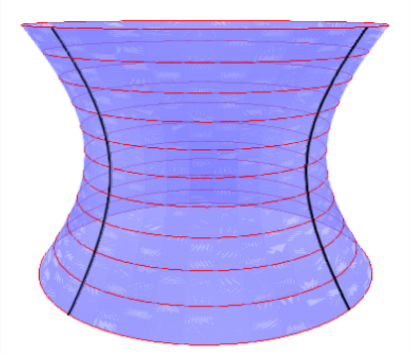

Verifying the foliation is measured.

The function $f : \alpha \times [0,1] \to \beta$ is
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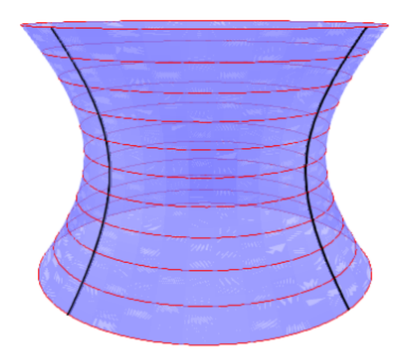

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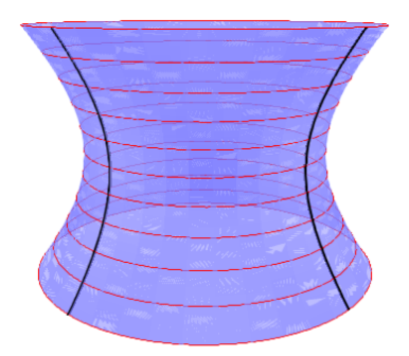

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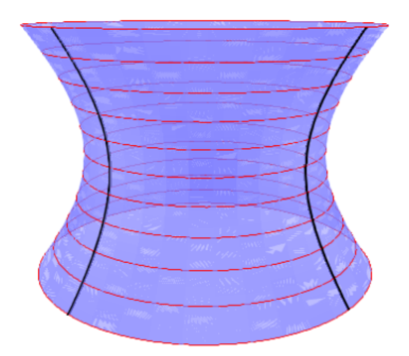

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We conclude that $(M, \mathcal{F})$ is a measured foliation.

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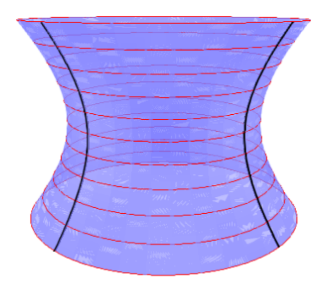

α is transverse to all of $\mathcal{F}$, and each leaf in $\mathcal{F}$ intersects α exactly once.

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- $\pi(t_1, (\pi(t_2, p_0)) = \pi(t_1 + t_2, p_0)$ (since two translations along $\mathbb{R}$ are easily composed).
- Therefore the definition of a dynamical system is satisfied by $(M, \mathbb{R}, \pi)$.

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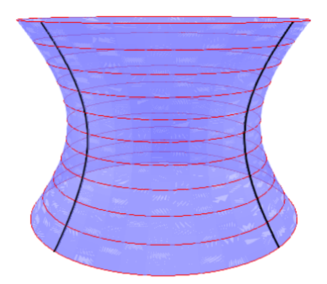
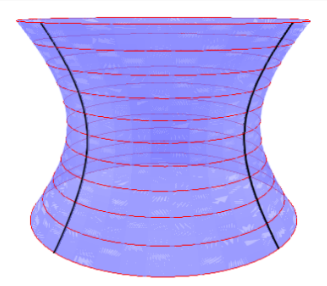
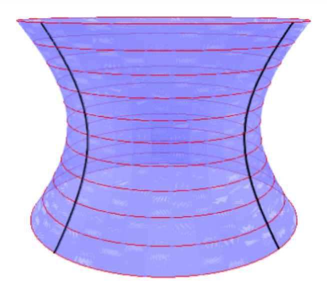




- Our chart maps preserve the orientation of leaves in the x direction, so $\pi^{-1}(t, A)$ will rotate A around M, scaled by the length of each leaf going through each point in A, so the change in θ between any two points in A is preserved under π^{-1}.



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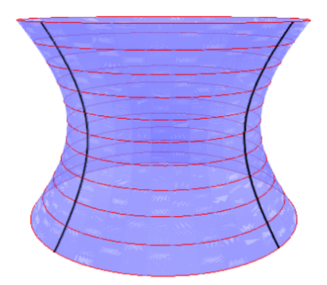

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- For any fixed angle of rotation around the hyperboloid, the metric on M is determined by r and z, and that distance is represented by changes in the x-coordinates under the chart map.
- The invariance of μ with respect to the x-direction after the chart map is applied, will preserve the change in the remaining components of ω_M under $\pi^{-1}(t, A)$.

- Therefore each point in A will have it's r and z coordinates preserved under $f^{-1}(t, A)$, and while each point will move a different distance, the change in the angle θ will be the same for each point. We conclude that ω_M is preserved under $\pi^{-1}(t, A)$ for any choice of $t \in \mathbb{R}$.

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- Let ν be the measure associated with ω_M, then, $(M, \mathbb{R}, f, \nu)$ is a measure-preserving dynamical system.

Following a similar construction

Theorem

An oriented, measured, 1-foliation $(\mathcal{F}, \mu)$ *on a compact, connected, orientable 3-manifold, yields a measure-preserving dynamical system.*

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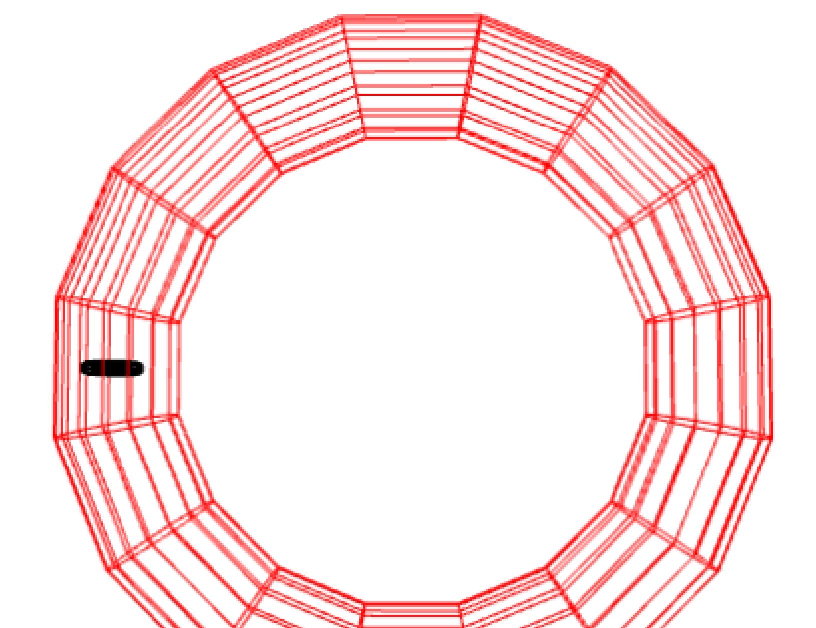
Back to the tori



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- Denote this as $M_{\sim f, l}$.

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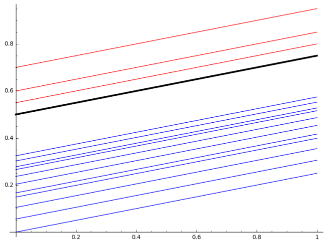
- Take the slanted suspension with slant $1/4$. The points 0 and 1 are fixed and $f(3/4) = 1/2$.

- There is a leaf from $(1/2, 0)$ to $(3/4, 1)$. This leaf comes from the slant, but the point $(3/4, 1)$ has the first coordinate shifted down by the same amount as the slant of the suspension. Thus $(3/4, 1)$ is identified with $(1/2, 0)$, creating a fixed circular leaf in the foliation.



Theorem

Let M be a 2 dimensional PL-manifold, μ a simplicial measure on M, and f a measure-preserving PL-homeomorphism on M. Then the foliation $M_{\sim,f,l}$ given by the slanted suspension of M under f, with slant l, is measured.

Flow bordisms and plugs

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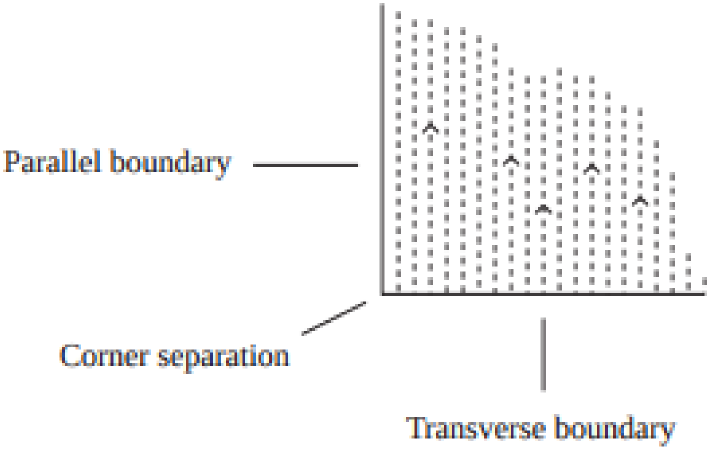
- The parallel boundary is a subset of the boundary of M such that the embedding of the foliation into the upper half-space of $\mathbb{R}^n$ is modeled by a foliation consisting entirely of horizontal lines.
- The transverse boundary is the subset where the same embedding consists entirely of vertical lines.
- A manifold may also be said to have *corners*, if the boundary of the manifold is piecewise of the same smoothness category as the manifold itself.


Parallel boundary
Corner separation
Transverse boundary

[GK]

Given a connected, compact manifold P, a *flow bordism* $\mathcal{P}$ is an oriented 1-foliation of P, such that all boundary of P is either transverse, parallel, or corners [GKK].

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Let F_- be the closure of the transverse boundary oriented inwards, and F_+ be the closure of the transverse boundary oriented outwards. We have two additional properties in which we are interested.

(i) There exists an infinite leaf with an endpoint in F_-

(ii) There exists a manifold F and two homeomorphisms $\alpha_- : F \to F_-$, $\alpha_+ : F \to F_+$ such that if $\alpha_+(p)$ and $\alpha_-(q)$ are endpoints of a leaf of $\mathcal{P}$, then $p = q$.

- If a flow bordism satisfies condition (i), but not condition (ii), it is a *semi-plug*.
- A flow bordism which satisfies (ii) but not (i) is an *un-plug*.
- If $\mathcal{P}$ has properties (i) and (ii), it is a *plug*.
- If $\mathcal{P}$ is a plug, the manifold P is the *support* of $\mathcal{P}$.

[GKK]

Modification of a foliation using a plug requires the operation of *insertion*. Let $\mathcal{P}$ be a flow bordism, and $\mathcal{F}$ a foliation on some manifold M.

- An *insertion map* is an embedding of $\mathcal{P}$, in which both F_- and F_+ are transverse to $\mathcal{F}$.
- A flow bordism $\mathcal{P}$ is *insertible* if there is an embedding of $\mathcal{P}$ into $\mathbb{R}^n$ which is transverse to vertical lines [GKK].

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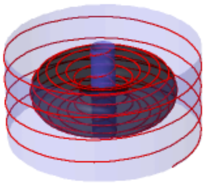
We are looking for a way to modify an existing foliation, by taking leaves whose image under the chart map is that of vertical lines in $\mathbb{R}^n$, and replacing a portion of these leaves with the leaves in our flow bordism.

A flow bordism with a torus in the middle

We will construct a slanted suspension that gives us a *PL* version of this.



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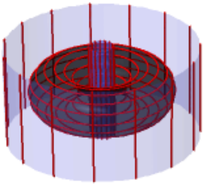
We are looking for a *PL* way to make this sort of transformation, to get the parallel boundary we need..



Theorem

Let $M = [a, b] \times [c, d]$, and $f : M \to M$ a PL-homeomorphism. Take the slanted suspension of M with slant l. If for all $p \in \partial M$, $f(p) \neq p - l$, then $M_{\sim,f,l}$ may be made into a flow bordism, via a leaf-preserving map $g : M_{\sim,f,l} \to M_{\sim,f,l}$. Furthermore, if $M_{\sim,f,l}$ is a measured-foliation, g may be chosen such that $g(M_{\sim,f,l})$ remains measured.

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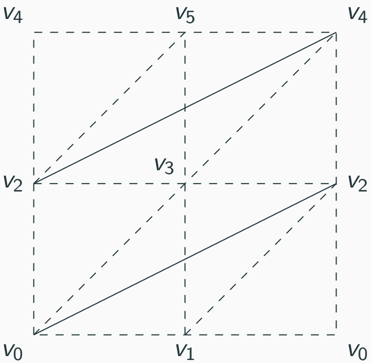
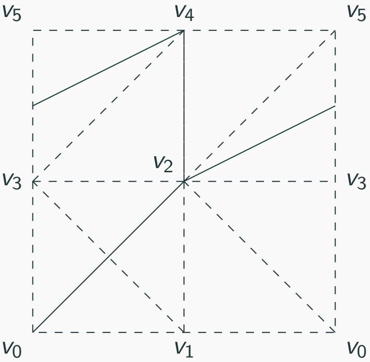
Corollary

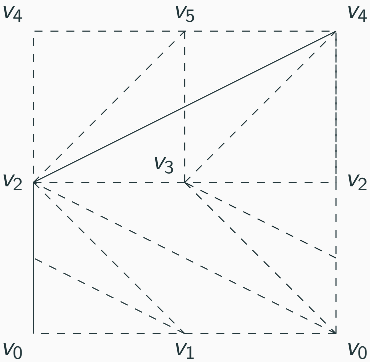
A PL flow bordism, with an invariant transverse measure, admits a PL dynamical system, which preserves measure.

Example





v_5
v_4
v_5
v_2
v_3
v_3
v_0
v_1
v_0


v4
v5
v4
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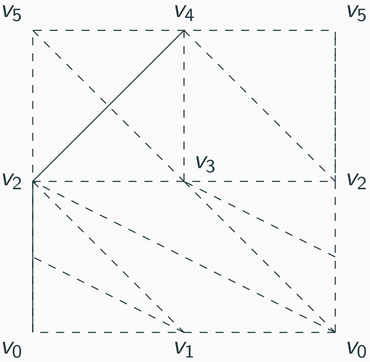
v_5
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v_0
v_1
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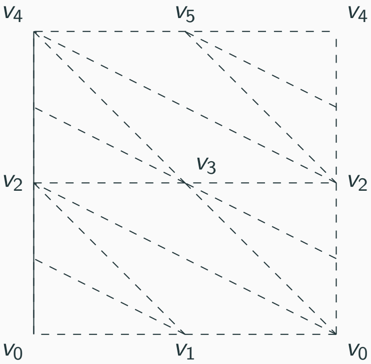
v_4
v_5
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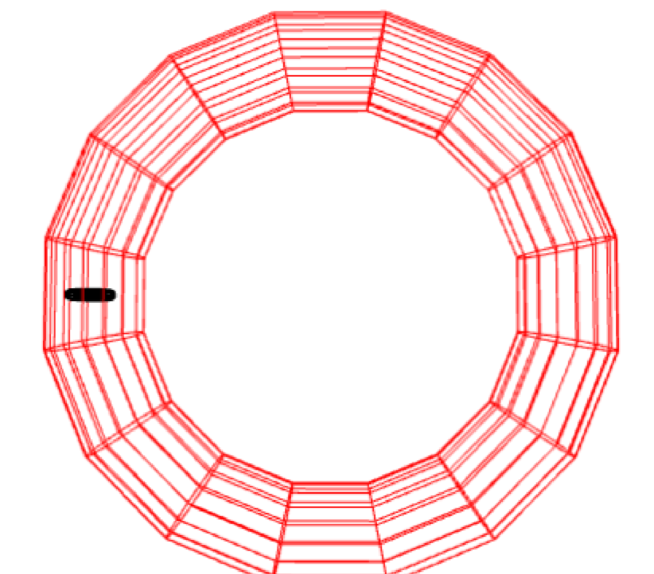

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Building our slanted suspension

Let $M = [-3,3] \times [-2,2]$. Triangulate M, and let $f : M \to M$ be the PL homeomorphism, which preserves area on 2-simplices, and shifts the vertices of the square $[-1,1] \times [-1,1]$ region down $1/4$.

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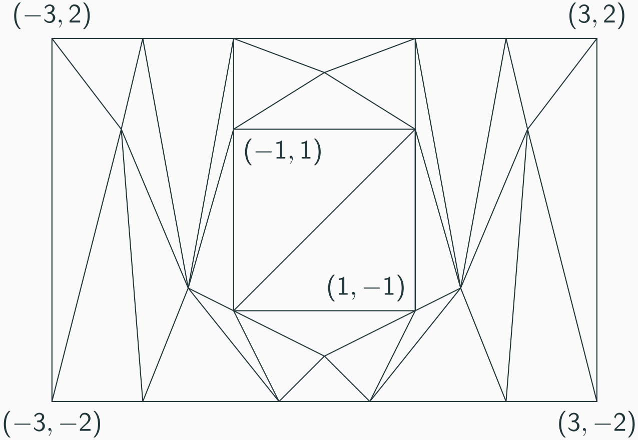

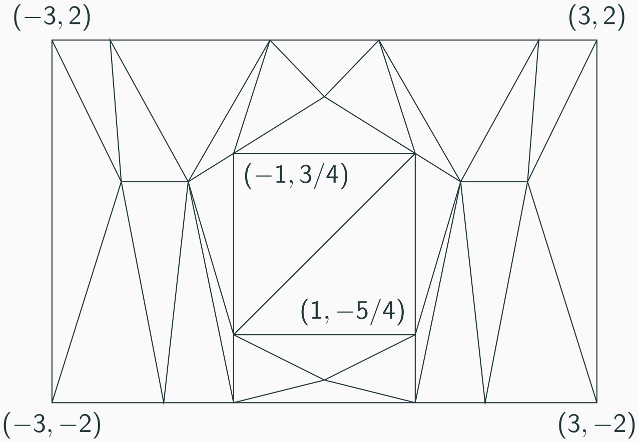
(−3, 2)
(3, 2)
(−1, 3/4)
(1, −5/4)
(−3, −2)
(3, −2)

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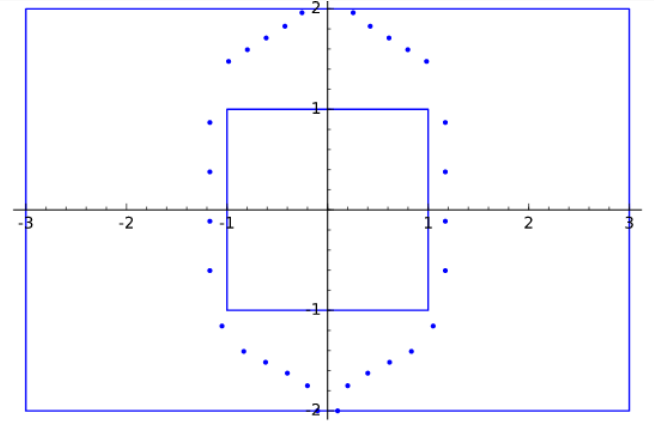
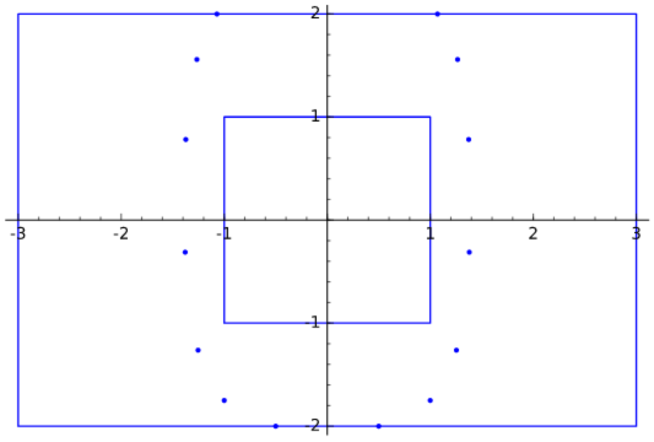
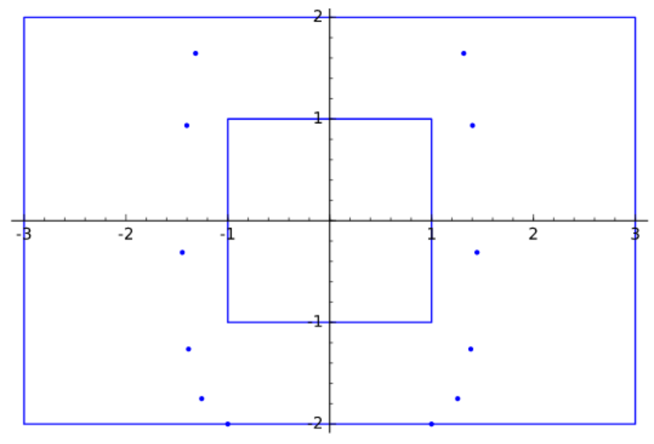
- We can use a Poincare return map to see where the leaves intersect M at each iteration.



Trajectories originating at (0.1,-2) and (-0.1,-2)



Trajectories originating at (1/2,-2) and (-1/2,-2)



Trajectories originating at (1,-2) and (-1,-2)

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- We were careful to construct our earlier foliation so that all of the leaves near each nested torus are vertical, so our flow bordism will insert nicely.

Dehn Surgery

- Begin with a 3-manifold, and pick a solid torus in it's interior.
- Identify a closed curve on the boundary of the torus, the *meridian*, which bounds a disc.

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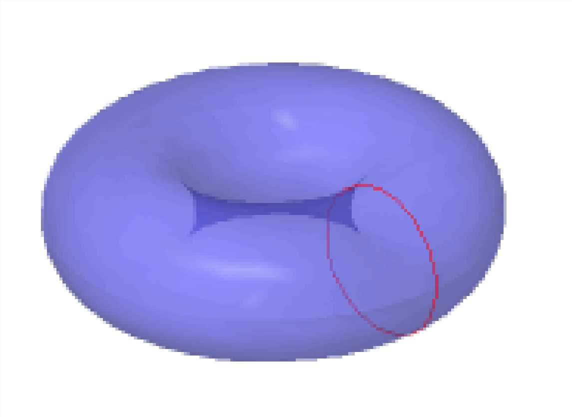

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Dehn Surgery

- Take another circle in the boundary of a torus, the *longitude*, which may or may not equal the meridian.
- By removing the torus from the 3-manifold, and gluing in a new one, with the longitude glued along the meridian, we effect a *Dehn Surgery* [G].

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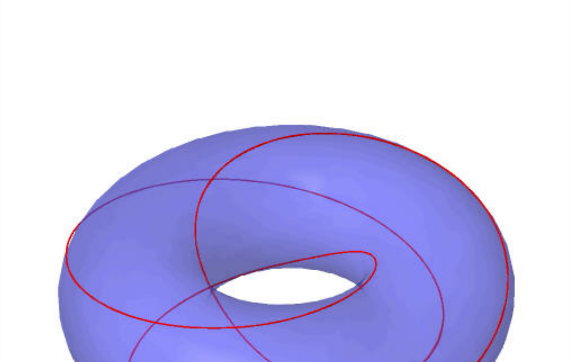

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- We therefore have a piecewise-linear, measure-preserving, non-singular dynamical system on $\mathbb{R}^3$, with each trajectory contained in a bounded set.

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